THE QUADRATIC FORMULA – RATIONAL SOLUTIONS

We now have all the tools needed to solve quadratic equations, whether they are factorable or not. In this chapter, our solutions will always be rational numbers; that is, no square roots in the final answers.

❑ *THE QUADRATIC FORMULA*

Here's the formula we created a couple of chapters back:

The quadratic equation $ax^2 + bx + c = 0$

has the solutions: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

❑ EXAMPLES OF USING THE QUADRATIC FORMULA

EXAMPLE 1: **Solve for x: $x^2 - 9x - 10 = 0$**

Solution: We see that this is a quadratic equation in standard form, where $a = 1$, $b = -9$, and $c = -10$. By the Quadratic Formula, x has (possibly) two solutions:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \quad x = \frac{-(-9) \pm \sqrt{(-9)^2 - 4(1)(-10)}}{2(1)}$$

$$\Rightarrow \quad x = \frac{9 \pm \sqrt{81 + 40}}{2}$$

$$\Rightarrow \quad x = \frac{9 \pm \sqrt{121}}{2}$$

$$\Rightarrow \quad x = \frac{9 \pm 11}{2}$$

Using the plus sign: $x = \frac{9+11}{2} = \frac{20}{2} = 10$

Using the minus sign: $x = \frac{9-11}{2} = \frac{-2}{2} = -1$

Final answer: $\boxed{x = 10, -1}$

Let's check these solutions:

Letting $x = 10$ in the original equation,

$$x^2 - 9x - 10 = \mathbf{10}^2 - 9(\mathbf{10}) - 10 = 100 - 90 - 10 = 0 \quad ✓$$

Letting $x = -1$,

$$x^2 - 9x - 10 = (\mathbf{-1})^2 - 9(\mathbf{-1}) - 10 = 1 + 9 - 10 = 0 \quad ✓$$

<u>EXAMPLE 2:</u> **Solve for n: $6n^2 + 40 = -31n$**

<u>Solution:</u> The first step is to transform this quadratic equation into standard form. Adding $31n$ to each side (and putting the $31n$ <u>between</u> the $6n^2$ and the 40) produces

$6n^2 + 31n + 40 = 0$ [It's now in standard form.]

Noting that $a = 6$, $b = 31$, and $c = 40$, we're ready to apply the Quadratic Formula:

$$n = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \quad n = \frac{-31 \pm \sqrt{(31)^2 - 4(6)(40)}}{2(6)}$$

$$\Rightarrow \quad n = \frac{-31 \pm \sqrt{961 - 960}}{12}$$

$$\Rightarrow \quad n = \frac{-31 \pm \sqrt{1}}{12}$$

$$\Rightarrow \quad n = \frac{-31 \pm 1}{12}$$

The two solutions are given by

$$\frac{-31+1}{12} = \frac{-30}{12} = -\frac{5}{2}$$

and $$\frac{-31-1}{12} = \frac{-32}{12} = -\frac{8}{3}$$

$$\boxed{n = -\frac{5}{2},\ -\frac{8}{3}}$$

<u>EXAMPLE 3:</u> **Solve for p: $2p^2 = 200$**

<u>Solution:</u> Bringing the 200 over to the left side of the equation gives us our quadratic equation in standard form:

$2p^2 - 200 = 0$

which, if it helps you, can be viewed as

$$2p^2 + \mathbf{0p} - 200 = 0$$

Notice that $a = 2$, $b = 0$, and $c = -200$. Substituting these values into the Quadratic Formula yields our two solutions for p:

$$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \quad p = \frac{-0 \pm \sqrt{(0)^2 - 4(2)(-200)}}{2(2)}$$

$$\Rightarrow \quad p = \frac{0 \pm \sqrt{0 - 4(2)(-200)}}{4}$$

$$\Rightarrow \quad p = \frac{\pm\sqrt{1600}}{4}$$

$$\Rightarrow \quad p = \frac{\pm 40}{4} = \pm 10$$

Thus, the solutions are $\boxed{p = \pm 10}$

Note: At or near the beginning of this problem, we could have divided each side of the equation by 2. The Quadratic Formula, although containing different values for *a* and *c*, would nevertheless have yielded the same answers, ±10. Try it.

EXAMPLE 4: **Solve for u: $5u^2 = 9u$**

Solution: First convert to standard form: $5u^2 - 9u = 0$, from which we deduce that $a = 5$, $b = -9$, and $c = 0$. Plugging these three values into the Quadratic Formula gives:

$$u = \frac{-(\mathbf{-9}) \pm \sqrt{(\mathbf{-9})^2 - 4(\mathbf{5})(\mathbf{0})}}{2(\mathbf{5})} = \frac{9 \pm \sqrt{81 - 0}}{10} = \frac{9 \pm 9}{10}$$

No radicals remain, so that issue is settled; using the plus and minus signs separately yields the two solutions:

$$\frac{9+9}{10} = \frac{18}{10} = \frac{9}{5} \quad \text{and} \quad \frac{9-9}{10} = \frac{0}{10} = 0$$

Final answer: $\boxed{u = \frac{9}{5},\ 0}$

EXAMPLE 5: **Solve for u: $28u - 4u^2 = 49$**

Solution: The standard quadratic form requires that the quadratic term (the squared variable) be in front, followed by the linear term, followed by the constant, all set equal to zero; that is, $ax^2 + bx + c = 0$. We achieve this goal with the following steps:

$$28u - 4u^2 = 49 \qquad \text{(the given equation)}$$

$$\Rightarrow \quad -4u^2 + 28u = 49 \qquad \text{(commutative property)}$$

$$\Rightarrow \quad -4u^2 + 28u - 49 = 0 \qquad \text{(subtract 49 from each side)}$$

At this point, it's in good form for the Quadratic Formula, but it's traditional to not allow the leading coefficient (the –4) to be negative. One way to change –4 into 4 is to multiply it by –1; but, of course, we will have to do this to each side of the equation.

$$\Rightarrow \quad \mathbf{-1}[-4u^2 + 28u - 49] = \mathbf{-1}[0] \qquad \text{(multiply each side by –1)}$$

$$\Rightarrow \quad 4u^2 - 28u + 49 = 0 \qquad \text{(distribute)}$$

When written this way,

$$4u^2 - 28u + 49 = 0,$$

and we deduce that $a = 4$, $b = -28$, and $c = 49$. On to the Quadratic Formula:

$$u = \frac{-(-28) \pm \sqrt{(-28)^2 - 4(4)(49)}}{2(4)}$$

$$= \frac{28 \pm \sqrt{784 - 784}}{8}$$

$$= \frac{28 \pm \sqrt{0}}{8} = \frac{28 \pm 0}{8} = \frac{28}{8} = \frac{7}{2}$$

And we're done: $\boxed{u = \frac{7}{2}}$

Notice that although all the previous quadratic equations had two solutions, this equation has just <u>one</u> solution. (Or does it have two solutions that are the same?)

<u>EXAMPLE 6</u>: **Solve for z: $z^2 + 3z + 3 = 0$**

<u>Solution</u>: This quadratic equation seems innocent enough, so let's take the values $a = 1$, $b = 3$, and $c = 3$ and place them in their proper places in the Quadratic Formula:

$$z = \frac{-(3) \pm \sqrt{(3)^2 - 4(1)(3)}}{2(1)} = \frac{-3 \pm \sqrt{9 - 12}}{2} = \frac{-3 \pm \sqrt{-3}}{2}$$

Houston . . . we have a problem. Under the assumption that you have not yet encountered imaginary numbers, we cannot consider the square root of a negative number at this point in your studies. If we try to take the square root of –3, the calculator will almost for sure indicate an error. We can't finish this calculation (at least not in the early and middle stages of Algebra), so we must agree that there is simply **NO SOLUTION** to the equation $z^2 + 3z + 3 = 0$.

Homework

1. Solve each quadratic equation:

a. $x^2 - 3x - 10 = 0$
b. $6n^2 + 19n - 7 = 0$
c. $d^2 + 7d = 0$
d. $2u^2 + u + 1 = 0$
e. $z^2 - 49 = 0$
f. $t^2 + 16 = 0$
g. $n^2 - 10n = 0$
h. $w^2 - 144 = 0$

2. Solve each quadratic equation:

a. $n^2 + 10n = -25$
b. $4x^2 = 4x - 1$
c. $3m^2 = 8m$
d. $x^2 = -5 + 3x$
e. $-x^2 - 5x + 6 = 0$
f. $2x^2 - 5x - 3 = 0$

3. Solve each quadratic equation:

a. $2w^2 - 70w + 500 = 0$
b. $w^2 + 2w + 5 = 0$
c. $9w^2 - 24w + 16 = 0$
d. $7n^2 = 10n$
e. $2x^2 - 15x + 7 = 0$
f. $15x^2 - 8 = 2x$
g. $4a^2 + 35a = -24$
h. $21t^2 = 20t + 9$

4. Solve each quadratic equation:

a. $w^2 - 12w + 35 = 0$
b. $2x^2 - 13x + 15 = 0$
c. $n^2 - 20n + 150 = 0$
d. $6a^2 - 31a + 40 = 0$
e. $-2x^2 - 10x + 28 = 0$
f. $25y^2 = 4$
g. $2z^2 + 2 = -4z$
h. $-4a^2 = 4a - 3$
i. $6u^2 = 47u + 8$
j. $0 = 4t^2 + 8t + 3$
k. $-n^2 + n + 56 = 0$
l. $-14x^2 = -40 + 46x$

Review Problems

5. Solve for y: $15y^2 + 2y - 8 = 0$

6. Solve for a: $-49a^2 - 9 = -42a$

7. Solve for x: $4x^2 = 1$

8. Solve for z: $10z^2 = 9z$

9. Solve for n: $3n^2 = 7n - 5$

10. Solve for w: $-2w^2 + 5w + 3 = 0$

❑ *To ∞ and Beyond!*

A. Solve for x: $\pi x^2 = \sqrt{2}\,x - \sqrt{\pi}$

B. Solve for x: $ax^2 + bx^2 + cx + dx + \pi^2 - \pi = 0$

Solutions

1. a. $5, -2$ b. $\frac{1}{3}, -\frac{7}{2}$ c. $0, -7$ d. No solution

e. ± 7 f. No solution g. $10, 0$ h. ± 12

2. a. -5 b. $\frac{1}{2}$ c. $0, \frac{8}{3}$

d. No solution e. $1, -6$ f. $3, -\frac{1}{2}$

3. a. 25, 10 b. No solution c. $\frac{4}{3}$ d. $0, \frac{10}{7}$

e. $7, \frac{1}{2}$ f. $-\frac{2}{3}, \frac{4}{5}$ g. $-8, -\frac{3}{4}$ h. $\frac{9}{7}, -\frac{1}{3}$

4. a. 5, 7 b. $5, \frac{3}{2}$ c. No solution d. $\frac{8}{3}, \frac{5}{2}$

e. 2, –7 f. $\pm\frac{2}{5}$ g. –1 h. $\frac{1}{2}, -\frac{3}{2}$

i. $8, -\frac{1}{6}$ j. $-\frac{1}{2}, -\frac{3}{2}$ k. 8, –7 l. $-4, \frac{5}{7}$

5. $\frac{2}{3}, -\frac{4}{5}$

6. $\frac{3}{7}$

7. $\pm\frac{1}{2}$

8. $0, \frac{9}{10}$

9. No solution

10. $3, -\frac{1}{2}$

"Education's purpose is to replace an empty mind with an open one."

Malcolm Forbes